COLUMBIA COLLEGE
MATHEMATICS 113
FINAL EXAMINATION
APRIL 2008
(3 HOURS)

Please note that the only electronic device that may be used during an examination is a non-graphing scientific calculator. Calculators may not be borrowed or lent during an examination.

1. Calculate the following limits. If a limit is infinite or does not exist you must explain why. You may not use L'Hôpital's rule for any of the limits. (4 marks each)

   a) \( \lim_{x \to 3} \frac{5 - \sqrt{x} + 22}{x^2 - 9} \)

   b) \( \lim_{x \to 2} \frac{x^2 + 10x - 24}{e^{x^2} - 1} \)

   c) \( \lim_{x \to 3} \frac{x^3 - 27}{x^2 + 5x - 24} \)

   d) \( \lim_{x \to 0} \frac{\sin(3x)}{5x} \)

   e) \( \lim_{x \to 2^+} \frac{x^4 + 6x^2 + 100}{x^2 - 5x + 6} \)

   f) \( \lim_{x \to \infty} \frac{\sqrt{9x^2 + 10x}}{9x + 4} \)

2. Suppose that the function \( f \) is defined using two formulas:
   
   If \( x \leq 3 \) then \( f(x) = \frac{x}{x - 4} \) and,
   
   If \( x > 3 \) then \( f(x) = 2x - 9 \)

   a) Prove that \( f \) is continuous at \( x = 3 \). (4)

   b) Explain why \( f \) is continuous at all other values of \( x \). (4)

3. Let \( f(x) = 8x^3 + 4x^2 \).
   
   a) Use the Intermediate Value theorem to prove that there is a number \( c \) in \([4, 6]\) such that \( f(c) = 810 \). (4)

   b) Use the Bisection method to find \( c \). Show all work. (4)

4. Give an "\( \varepsilon, \delta \)" proof for \( \lim_{x \to 3} 2x^2 - 7x = -3 \). (6)

5. a) Using only the definition of the derivative function, find the derivative of \( f(x) = x + \frac{2}{x} \). (4)
b) Using the result of part a), find the slope-intercept equation of the
tangent line to the curve \( y = x + \frac{2}{x} \) at the point \( (2,3) \). (4)

6. Suppose that the function \( f(x) \) is defined using two formulas:
   
   If \( x < 2 \) then \( f(x) = 3x^2 + x + 2 \) and
   
   if \( x \geq 2 \) then \( f(x) = 11x - 6 \).

   Prove that \( f \) is not differentiable at \( x = 2 \). (4)

7. Find the derivatives of the following functions. You do not have to
   simplify your answers. (4 marks each)
   
   a) \( \arctan(6x + 1) \)  
   b) \( \sqrt{\tan(6x + 1)} \)  
   c) \( x^3 \ln(x^3 + 1) \)

8. Use implicit differentiation to find \( \frac{dy}{dx} \) in terms of \( x \) and \( y \) given the
   relation \( xy^3 + 2x^2y^2 + 5x + 3y = 27 \) and then find the slope-intercept
   equation of the line that is tangent to the graph of this relation at the point
   \( (1,2) \). (6)

9. A 10-foot ladder is leaning against a wall so that the height of the ladder
   is \( y \) feet above the ground. Let \( \theta \) be the angle formed by the ladder and the
   ground. Suppose that the top of the ladder is sliding down the wall at a
   constant rate of 2 feet per second. At what rate is the angle \( \theta \) changing when
   \( y = 6 \) feet? State the answer in degrees per second. (6)
10. Let \( f(x) = \frac{x^2 + x}{\sqrt{x} + 1} \) be restricted to the interval \((0, \infty)\).

a) Prove that \( f(x) \) has a well-defined inverse \( g(x) = f^{-1}(x) \). (4)

b) Using the theorem for the derivative of an inverse function, find \( g'(6) \). (4)

11. Estimate \( \sec\left(\frac{\pi}{4} + \frac{1}{10}\right) \) using the linear approximation of \( f(x) = \sec(x) \) for values of \( x \) near \( a = \frac{\pi}{4} \). First give the exact value of the answer and then give it in decimal form rounded to six decimal places. Also give the calculator value rounded to six decimal places. (4)

12. Use logarithmic differentiation to find the derivative of
\[ y = \frac{(2x^2 + 1 \ln x)\sqrt{x^2 + 3}}{e^{x^2 + 1}}. \]
You must use the properties of logarithms as much as possible and \( y' \) must be given in terms of \( x \) only. (4)

13. Let \( f(x) = x(\ln x)^2 \). Using a sign chart, find the intervals where \( f \) is increasing and the intervals where \( f \) is decreasing. Also give both coordinates of any relative extrema. (4)

14. Let \( f(x) = \frac{1}{x^2 + 3} \). Using a sign chart, find the intervals where \( f \) is concave up and the intervals where \( f \) is concave down. Also give both coordinates of any inflection points. (4)

15. Let \( f(x) = \frac{8}{x} + \frac{16}{x^2} \). State the intercepts and asymptotes of the graph of \( f \). Using a sign chart, find where \( f \) is increasing and where \( f \) is decreasing and give both coordinates of any local extrema. Using a sign chart, find the intervals where \( f \) is concave up and where \( f \) is concave down and give both coordinates of any inflection points. Accurately draw the graph indicating all these features. (10)
16. Accurately draw the graph of a function \( f(x) \) that has all the following properties. Indicate any local extrema or asymptotes.

- \( f \) is continuous on \( \mathbb{R} \)
- \( f(-2) = -1 \) and \( f'(-2) = 0 \)
- \( f(0) = 0 \) and \( f'(0) = 1 \)
- on the interval \((-\infty, -2)\), \( f'(x) < 0 \) and \( f''(x) > 0 \)
- on the interval \((-2, 0)\), \( f'(x) > 0 \) and \( f''(x) > 0 \)
- on the interval \((0, 2)\), \( f'(x) > 0 \) and \( f''(x) < 0 \)
- on the interval \((2, 4)\), \( f'(x) < 0 \) and \( f''(x) < 0 \)
- on the interval \((4, \infty)\), \( f'(x) > 0 \) and \( f''(x) < 0 \)
- \( \lim_{x \to \infty} f(x) = 0 \)

17. The function \( f(x) = \frac{1}{2}x^2 - \sin x \) has a critical point \( c \) in the interval \((0, 2)\). Use Newton's method to find \( c \) rounded to two decimal places. (6)

18. Use L'Hôpital's rule to find the following limits: (4 marks each)

a) \( \lim_{x \to 0} \frac{1 - \cos x}{x^2} \)

b) \( \lim_{x \to \infty} \frac{(1 + 3x)^{1/3}}{\ln x} \)

19. Let \( f(x) = \frac{x^2 - 9}{x - 1} \)

a) Use the Mean Value Theorem to prove that there is a number \( c \) in the interval \((2, 5)\) such that \( f'(c) = 3 \). (4)

b) Find the exact value of \( c \). (4)

20. A triangle with one side on the x-axis is inscribed in the triangle shown above. Find, with proof, the maximum area of such a triangle. (4)

Total = 150
Answers.
1. a) \[ \lim_{x \to 3} \frac{5 - \sqrt{x + 22}}{x^2 - 9} = \lim_{x \to 3} \frac{(5 - \sqrt{x + 22})(5 + \sqrt{x + 22})}{(x - 3)(x + 3)(5 + \sqrt{x + 22})} \]
\[ = \lim_{x \to 3} \frac{1}{(x - 3)(5 + \sqrt{x + 22})} = \frac{1}{60}. \] (4)

b) Let \( y = x - 2 \). Then \[ \lim_{x \to 2} \frac{x - 2}{e^{x - 2} - 1} = \lim_{y \to 0} \frac{y}{e^{y} - 1} = 1 \] (theorem)

Therefore \[ \lim_{x \to 2} \frac{x^2 + 10x - 24}{e^{x - 2} - 1} = \lim_{x \to 2} \frac{x - 2}{e^{x - 2} - 1} \lim_{x \to 12} x = 1 \times 14 = 14 \] (4)

c) \[ \lim_{x \to 3} \frac{x^3 - 27}{x^2 + 5x - 24} = \lim_{x \to 3} \frac{(x - 3)(x^2 + 3x + 9)}{(x - 3)(x + 8)} = \lim_{x \to 3} \frac{x^2 + 3x + 9}{x + 8} \]
\[ = \frac{27}{11}. \] (4)

d) Let \( y = 3x \). Then \[ \lim_{x \to 0} \frac{\sin(3x)}{x} = \lim_{y \to 0} \frac{\sin(y)}{y} = 3 \lim_{y \to 0} \frac{\sin(y)}{y} = 3. \]

Therefore \[ \lim_{x \to 0} \frac{\sin(3x)}{5x} = \lim_{x \to 0} \frac{\sin(3x)}{x} \times \frac{3}{5} = 3. \] (4)

e) \[ \lim_{x \to 0^+} \frac{x^4 + 6x^2 + 100}{x^5 - 5x + 6} = \lim_{x \to 0^+} \frac{x^4 + 6x^2 + 100}{(x - 2)(x - 3)} \]
is close to 2 and bigger than 2 then the top is close to the positive number 140 and the bottom is a negative number close to 0. (4)

f) \[ \lim_{x \to \infty} \frac{9x^2 + 10x}{9x + 4} = \lim_{x \to \infty} \frac{9x + 10}{9 + 4/x} = \frac{1}{3} \] because as \( x \to \infty \),
\[ 10/x \to 0 \] and \( 4/x \to 0 \). (4)

2. a) \( f \) is continuous at \( x = 3 \) because all three conditions of continuity are satisfied:
1. \( f(3) = -3 \) by the first formula
2. \( \lim f(x) \) exists because: \( \lim f(x) = \lim_{x \to 3} x = -3 \) and
\[ \lim f(x) = \lim_{x \to 3^-} x = -3 \] and:
\[ \lim f(x) = \lim_{x \to 3^+} x = -3 \] and:
3. \( \lim f(x) = f(3) \) (4)
2. b) \( f \) is continuous on \((-\infty, 3)\) because the rational function \( \frac{x}{x-4} \) is continuous at all real numbers except \( x = 4 \), and \( f \) is continuous on \((3, \infty)\) because the polynomial \( 2x - 9 \) is continuous at all real numbers. (4)

3. a) \( f(x) = 8x^3 + 4x^2 \) is a polynomial and therefore is continuous on the closed interval \([4, 6]\). Since \( f(4) = 576 < 810 < 1872 = f(6) \), by the Intermediate Value Theorem there is a number \( c \) in \((4, 6)\) such that \( f(c) = 810 \). (4)

b) The midpoint of \([4, 6]\) is 5 and \( f(5) = 1100 \). Therefore \( c \) is in \([4, 5]\). The midpoint of \([4, 5]\) is 4.5 and \( f(4.5) = 810 \). Therefore \( c = 810 \). (4)

4. \( \lim_{x \to 3} 2x^2 - 7x = -3 \)

Proof: Let \( \varepsilon \) be a given positive number. \( |2x^2 - 7x - (-3)| = |2x - 1| |x - 3| \)

If, for example, \( |x - 3| < 1 \) then \( 2 < x < 4 \), \( 3 < 2x - 1 < 7 \) and \( |2x - 1| < 7 \)

Then \( |2x^2 - 7x - (-3)| = |2x - 1| |x - 3| < 7|x - 3| \) if \( |x - 3| < 1 \) and

\[ 7|x - 3| < \varepsilon \text{ if } |x - 3| < \frac{\varepsilon}{7} \]

Therefore if we choose \( \delta \) to be the smaller of the two positive numbers 1 and \( \frac{\varepsilon}{7} \), then we have that \( |2x^2 - 7x - (-3)| < \varepsilon \) if \( 0 < |x - 3| < \delta \). (6)

5. a) \( f(x) = x + \frac{2}{x} \rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \)

\[ \frac{x + h + \frac{2}{x + h} - x - \frac{2}{x}}{h} = \lim_{h \to 0} \frac{h + 2x - 2(x + h)}{x(x + h)} = \lim_{h \to 0} \frac{1 - \frac{2}{x}}{x(x + h)} = 1 - \frac{2}{x^2}. \] (4)

b) At \( x = 2 \), the slope of the tangent line is \( f'(2) = 1 - \frac{2}{4} = \frac{1}{2} \). Therefore the equation of the tangent line at \((2, 3)\) is of the form \( y = \frac{1}{2}x + C \) for some...
constant C. Since (2,3) is a solution, 3 = 1 + C \rightarrow C = 2$. Therefore the equation of the tangent line at (2,3) is $y = \frac{1}{2} x + 2$. (4)

6. $\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{3(2+h)^2 + (2+h) + 2 - 16}{h}$

$= \lim_{h \to 0^-} \frac{13h + 3h^2}{h} = \lim_{h \to 0^-} 13 + 3h = 13$ but:

$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{11(2+h) - 6 - 16}{h} = \lim_{h \to 0^+} \frac{11h}{h} = 11$.

Therefore the two-sided limit $f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$ does not exist which means that $f$ is not differentiable at $x = 2$. (4)

7. a) $\frac{d}{dx} \arctan(6x+1) = \frac{6}{1 + (6x + 1)^2}$ (4)

b) $\frac{d}{dx} \sqrt{\tan(6x+1)} = \frac{6}{2\sqrt{\tan(6x+1)}} \sec^2(6x + 1)$ (4)

c) $\frac{d}{dx} \ln(x^3 + 1) = 3x^2 \ln(x^3 + 1) + x^3 \left( \frac{3x^2}{x^3 + 1} \right)$ (4)

8. $\frac{d}{dx} xy^3 + 2x^2 y^2 + 5x + 3y = \frac{d}{dx} 27,

y^3 + 3xy^2 y' + 4xy^2 + 4x^2 yy' + 5 + 3y' = 0, \quad y' = \frac{-y^3 - 4xy^2 - 5}{3xy^2 + 4x^2 y + 3}$

At (1,2), $y' = \frac{-8 - 16 - 5}{12 + 8 + 3} = -\frac{29}{23}$. Therefore the equation of the tangent line at (1,2) is of the form $y = -\frac{29}{23} x + C$ for some constant $C$. Since (1,2) is a solution, $x = -\frac{29}{23} + C \rightarrow C = \frac{75}{23}$ and the equation of the tangent line is
\[ y = -\frac{29}{23} x + \frac{75}{23}. \quad (6) \]

9. Given that \( \theta = \text{Arccos}(y/10) \) and \( \frac{dy}{dt} = -2 \) (ft/sec) we have:

\[
\frac{d\theta}{dt} = \frac{\frac{dy}{dt}}{\frac{dy}{d\theta}} = \frac{1}{10\sqrt{1-y^2/100}} (-2) = -\frac{1}{5\sqrt{1-y^2/100}} \sqrt{100-y^2}.
\]

Therefore at \( y = 6, \frac{d\theta}{dt} = -\frac{1}{4} \) radians/sec. Since one radian = \( \frac{180}{\pi} \) \( \theta \) is changing at a rate of \(-\frac{45}{\pi}\) degrees per second. (6)

10. a) \( f(x) = \frac{x^2 + x}{\sqrt{x+1}} \implies f'(x) = \frac{(2x+1)\sqrt{x+1} - (x^2 + x)}{2(x+1)} \)

\[ = \frac{2(2x+1)(x+1)-x^2-x}{2(x+1)^{3/2}} = \frac{3x^2 + 5x + 2}{2(x+1)^{3/2}} \]

which is positive-valued for all \( x > 0 \). Therefore \( f \) is strictly increasing on \((0,\infty)\) and therefore is also one-to-one on \((0,\infty)\) and therefore has an inverse \( g(x) = f^{-1}(x) \). (4)

b) Since \( f(3) = 6, g(6) = 3 \) and \( g'(6) = \frac{1}{f'(3)} = \frac{1}{11/4} = \frac{4}{11}. \) (4)

11. With \( f(x) = \sec(x) \) and \( a = \frac{\pi}{4}, \ f'(x) = \sec(x) \tan(x), \ f'(\frac{\pi}{4}) = \sqrt{2} \times 1 = \sqrt{2}. \)

Therefore using linear approximation

\[ \sec\left(\frac{\pi}{4} + \frac{1}{10}\right) \approx \sec\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right)\left(\frac{\pi}{4} + \frac{1}{10}\right) \approx \sqrt{2} + \frac{11\sqrt{2}}{10} \approx 1.55635 \]

and the calculator value of \( \sec\left(\frac{\pi}{4} + \frac{1}{10}\right) \) is 1.579825. (4)

12. Taking the natural logarithm of both sides of \( y = \frac{(2x^2 + \ln x)\sqrt{x^2 + 3}}{e^{x^2 + 1}} \):
\[
\ln y = \ln(2x^2 + \ln x) + \frac{1}{2}\ln(x^2 + 3) - (x^2 + 1). \text{ Differentiating both sides with respect to } x:
\]
\[
y' = \frac{4x + 1/x}{2(2x^2 + \ln x)} + \frac{x}{x^2 + 3} - 2x. \text{ Therefore}
\]
\[
y' = \left(\frac{4x + 1/x}{2(2x^2 + \ln x)} + \frac{x}{x^2 + 3} - 2x\right) (4)
\]

13. \(f(x) = x(\ln x)^2 \rightarrow f'(x) = (\ln x)^2 + x \left(\frac{2\ln x}{x}\right) = (\ln x)^2 + 2\ln x\)

\[
= \ln x(\ln x + 2). \text{ Therefore } f \text{ has two critical points: } x = 1 \text{ and } x = e^{-2}
\]

\[
\begin{array}{c|cccc}
 x & 0 & e^{-2} & 1 & \\
\hline
 f'(x) & - & 0 & - & + \\
\end{array}
\]

\(f\) is increasing on \((0, e^{-2})\) and on \((1, \infty)\) and is increasing on \((e^{-2}, 1)\) so that \((e^{-2}, 4e^{-2})\) is a relative maximum and \((1,0)\) is a relative minimum. (4)

14. \(f(x) = \frac{1}{x^2 + 3} \rightarrow f'(x) = \frac{-2x}{(x^2 + 3)^2}\).

\[
f''(x) = \frac{-2(x^2 + 3)^2 + 2x(4x(x^2 + 3))}{(x^2 + 3)^4} = \frac{6x^2 - 6}{(x^2 + 3)^3}
\]

\[
\begin{array}{c|cccc}
 x & - & 1 & 1 & + \\
\hline
 f''(x) & + & 0 & - & + \\
\end{array}
\]

\(f\) is concave up on \((-\infty, -1)\) and on \((1, \infty)\) and concave down on \((-1,1)\) so that \((-1, \frac{1}{4})\) and \((1, \frac{1}{4})\) are inflection points. (4)

15. \(f(x) = \frac{8}{x} + \frac{16}{x^2}\) has two asymptotes: the \(x\)-axis and the \(y\)-axis. (1)

The graph has no \(y\)-intercept \(\frac{8}{x} + \frac{16}{x^2} = 0 \text{ if } 16x = -8x^2 \rightarrow x = -2.\)

Therefore the graph has one \(x\)-intercept: \((-2, 0)\) (1)
\[ f'(x) = -\frac{8}{x^2} - \frac{32}{x^3} = \frac{-8x - 32}{x^3} \]

\[
\begin{array}{c|c|c|c}
 x & -4 & 0 & \text{und} \\
 f'(x) & - & 0 & + \\
\end{array}
\]

\[ f \text{ is decreasing on } (-\infty, -4) \text{ and on } (0, \infty) \text{ and is increasing on } (-4, 0) \text{ so that } (-2, -1) \text{ is a relative minimum.} \]

\[ f''(x) = \frac{16}{x^3} + \frac{96}{x^4} = \frac{16x + 96}{x^4} \]

\[
\begin{array}{c|c|c|c}
 x & -6 & 0 & \text{und} \\
 f''(x) & - & 0 & + \\
\end{array}
\]

\[ f \text{ is concave down on } (-\infty, -6) \text{ and is concave up on } (-6, 0) \text{ and } (0, \infty) \text{ so that } (-6, \frac{4}{9}) \text{ is an inflection point.} \]

16.

\[ A(-6, -\frac{4}{9}) \text{ is an inflection point.} \]
\[ B(-2, -1) \text{ is a relative minimum.} \]

\[ \text{rel. max at } x = 2 \]
\[ \text{rel. min at } x = -1 \]
\[ (0, 0) \text{ IP} \]
\[ \text{rel. min at } x = 4 \]
\[ \text{the } x\text{-axis is a horizontal asymptote} \]
17. \( f(x) = \frac{1}{2}x^2 - \sin x \rightarrow f'(x) = x - \cos x \). From the graphs \( y = x \) and \( y = \cos x \) we see that there is exactly one zero of \( f(x) \) in \((0,2)\):

![Graph of \( y = x \) and \( y = \cos x \)]

Using Newton’s method for \( g(x) = x - \cos x \) and \( x_0 = 0.75 \) we have that

\[
x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}
\]

so that \( x_1 \approx 0.739111 \) and \( x_2 \approx 0.739085 \) from which we conclude that the critical point of \( f \) rounded to two decimal places is 0.74. (6)

18. a) \( \lim_{x \to 0} \frac{1 - \cos x}{x^2} \) is an indeterminate form of the type \( \frac{0}{0} \). Since the top and bottom functions are differentiable in any interval containing 0, we have by L’Hôpital’s rule that

\[
\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{2}.
\]

(4)

b) We first find the limit \( \lim_{x \to \infty} \ln(1 + 3x)^{\frac{1}{\ln x}} \) which is an indeterminate form of the type \( \frac{\infty}{\infty} \). Since the top and bottom functions are differentiable on \((0,\infty)\), we have by L’Hôpital’s rule that

\[
\lim_{x \to \infty} \frac{\ln(1 + 3x)}{\ln x} = \lim_{x \to \infty} \frac{3/(1 + 3x)}{1/x} = \lim_{x \to \infty} \frac{3}{1 + 3x} = \lim_{x \to \infty} \frac{3}{3 + 1/x} = 1.
\]

Therefore the given limit, \( \lim_{x \to \infty} (1 + 3x)^{\frac{1}{\ln x}} \), is \( e \). (4)
19. a) \( f(x) = \frac{x^2 - 9}{x - 1} \) is a rational function and therefore is continuous at all values of \( x \) except at \( x = 1 \) where it is undefined. Therefore \( f \) is continuous on \([2,5] \). \( f'(x) = \frac{2x(x - 1) - (x^2 - 9)}{(x - 1)^2} = \frac{x^2 - 2x + 9}{(x - 1)^2} \) exists for all values of \( x \) except \( x = 1 \). Therefore \( f \) is differentiable on \((2,5)\). Since both conditions of the Mean Value Theorem have been satisfied for \( f \) on the interval \([2,5] \), it follows that there is a number \( c \) in \((2,5) \) such that

\[
f'(c) = \frac{f(5) - f(2)}{5 - 2} = \frac{4 - (-5)}{3} = 3. \quad (4)
\]

b) \( f'(c) = 3 \rightarrow \frac{c^2 - 2c + 9}{(c - 1)^2} = 3 \rightarrow c^2 - 2c + 9 = 3c^2 - 6c + 3 \) or

\[
c^2 - 2c + 3 = 0, (c - 3)(c + 1) = 0 \rightarrow c = 3. \quad (4)
\]

20.

In the diagram, point \( A \) is on the line \( y = x \) and point \( B \) is on the line \( y = -\frac{1}{2}x + \frac{3}{2} \). Therefore, if we let \( z \) be the \( x \)-coordinate of \( B \) then \( 1 < z < 3 \) and the \( y \)-coordinates of both \( B \) and \( A \) are \(-\frac{1}{2}z + \frac{3}{2} \) which means that the \( x \)-coordinate of \( A \) is also \(-\frac{1}{2}z + \frac{3}{2} \). The height of the rectangle is \(-\frac{1}{2}z + \frac{3}{2} \) and the width of the rectangle is \( z - \left(-\frac{1}{2}z + \frac{3}{2}\right) = \frac{3}{2}z - \frac{3}{2} \). The area of the rectangle is

\[
A(z) = \left(-\frac{1}{2}z + \frac{3}{2}\right)\left(\frac{3}{2}z - \frac{3}{2}\right) = -\frac{3}{4}z^2 + 3z - \frac{9}{4} \quad \text{where} \quad 1 < z < 3.
\]

\( A'(z) = -\frac{3}{2}z + 3 \) which equals zero at \( z = 2 \).
Since $A$ is increasing on $(1,2)$ and decreasing on $(2,3)$, the maximum value of $A$ is $A(2) = \frac{3}{4}$. (4)