COLUMBIA COLLEGE
Mathematics 113
Final Examination April, 2006

1. a) State the informal definition for \( \lim_{x \to a} f(x) = L \).

b) State the "\( \epsilon, \delta \)" definition for \( \lim_{x \to a} f(x) = L \).

c) Give an "\( \epsilon, \delta \)" proof for \( \lim_{x \to 3} 2x^3 + x = 57 \)

2. Let the function \( f \) be defined using two formulas: If \( x \leq 3 \) then 
\( f(x) = \frac{7x + 3}{x^2 - 1} \) and if \( x > 3 \) then \( f(x) = \sqrt{2x + 3} \)

a) Prove that \( f \) is continuous at \( x = 3 \). You must verify that the three 
conditions of the definition have been satisfied.

b) Find all the other values of \( x \) where \( f \) is continuous. You must use 
theorems to justify your answer.

3. Find the following limits without using L'Hôpital's rule. Possible answers 
are: a finite real number, \( \infty \), \( -\infty \), or "does not exist". Show all work. (4 
marks each)

a) \( \lim_{x \to 3} \frac{2x^3 - 10x - 24}{x^2 + 2x - 15} \)

b) \( \lim_{x \to 3} \frac{\cos(\pi x)}{9 - x^2} \)

c) \( \lim_{x \to 3} \frac{7 - \sqrt{3x + 40}}{3 - \sqrt{2x + 3}} \)

d) \( \lim_{x \to \infty} \frac{(x + 2)^3 \sqrt{8x^6 + 1000}}{x^3 + x^2 + 12} \)

e) \( \lim_{x \to 0} \frac{\sin(2x)}{e^{3x} - 1} \)
4. During the first 10 seconds of a bicycle trip, a bicycle travels along a straight road so that in \( t \) seconds it travels \( s(t) = \frac{1}{2}t^2 + 50t \) feet.

a) How far does the bicycle travel in the first 10 seconds? (1)
b) What is the average velocity of the bicycle during the first 10 seconds? (2)
c) By computing a limit, find the velocity of the bicycle at \( t = 5 \)? (3) (The appropriate units must be used in the answers for a), b) and c))

5. Using only the definition of the derivative function \( f'(x) \), find \( f'(x) \) for the function \( f(x) = 2x + \frac{1}{3x} \). (4)

6. Let \( f(x) = \begin{cases} 9 - x^2 & \text{if } x < 3 \\ x - 3 & \text{if } x \geq 3 \end{cases} \)
a) Prove (by showing that a certain two-sided limit does not exist) that \( f \) is not differentiable at \( x = 3 \). (3)
b) Accurately draw the graph of \( f \) on the interval \([0,6]\) and explain how this graph shows that \( f \) is not differentiable at \( x = 3 \). (2)

7. Find the slope-intercept equation of the line that is tangent to the graph \( y = \frac{x\sqrt{x} + 2}{x^2 + x + 2} \) at the point \((2, \frac{1}{2})\). (4)

8. Find \( \frac{d^2y}{dx^2} \) (in simplified form) for \( y = x\sin(2x) + 2x\cos x \). (4)
9. In the diagram to the right, the units of length are feet. The angle \( \theta \) is formed by the x-axis and the line from the origin to a point \( P \) that is moving along the line \( y = 2x - 2 \) so that the x-coordinate of \( P \) is changing at a rate of \( \frac{dx}{dt} = 2 \) ft/sec. At what rate is the angle \( \theta \) changing when \( x = 2 \)? (4)

10. 

a) The graph of the relation \( x^4 + y^2 - 2x^2y + 5x^2 - 4y + 3 = 0 \) is shown above. Use implicit differentiation to express \( \frac{dy}{dx} \) as a function of \( x \) and \( y \) and then find the slope-intercept equation of the line that is tangent to the graph at the point \( \left( \frac{\sqrt{3}}{2}, \frac{13}{4} \right) \). (4)

b) There appear to be points on the graph where the tangent lines are vertical. Identify these points and verify that the tangent lines at these points are indeed vertical. (2)
11. Let \( f(x) = x^6 - 16x^3 + 66 \) be restricted to the domain \( \{ x : x \geq 2 \} \)

a) Prove that \( f \) is a one-to-one function. (2)
b) Using the theorem for the derivatives of inverse functions, find the value of the derivative of the inverse of \( f \) at \( x = 363 \). (2)
c) Find a formula for \( f^{-1}(x) \) and then differentiate \( f^{-1}(x) \) directly to find the answer for part b). (4)

12. Use L'Hôpital's rule to find:

a) \( \lim_{x \to 0^+} x \ln x \) (3)  
   b) \( \lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{\ln(x+1)} \right) \) (4)  
   c) \( \lim_{x \to 0} (\cos x)^{1/x^2} \) (4)

13. Suppose that \( f \) is a function that satisfies all the following conditions:

- \( f \) is continuous on \([0,6]\) and on \((6,\infty)\)
- \( f(0) = 1, f(2) = 4, f(4) = 2, f(8) = 1, f(10) = 1/2 \)
- On \((0,2)\): \( f'(x) > 0 \) and \( f''(x) > 0 \)
- On \((2,4)\): \( f'(x) < 0 \) and \( f''(x) > 0 \) and \( f'(4) = 0 \)
- On \((4,6)\): \( f'(x) > 0 \)
- \( \lim_{x \to 6^-} f(x) = \infty, \lim_{x \to 6^+} f(x) = -\infty \)
- \( f'(8) = 0 \)
- On \((8,10)\): \( f''(x) < 0 \) and \( f'(x) < 0 \)
- On \((10,\infty)\): \( f''(x) > 0 \)
- \( \lim_{x \to \infty} f(x) = 0 \)

Accurately draw the graph of \( f \) on the interval \([0,12]\). All relative extrema, inflection points and asymptotes must be identified in the graph. (6)

14. Use logarithmic differentiation to find the derivative of

\[
y = \frac{\sin^3(2x+1)\sqrt{2x+9}}{(x^2 + x + 1)^4}.
\] (4)

15.a) State the Extreme Value Theorem and explain how to find the absolute extrema of a function that satisfies the conditions of the Extreme Value Theorem. (3)
15 b) Find the absolute extrema of \( f(x) = \cos(x) + \cos(2x) \) on \([0, 2\pi]\) (4)

16. Let \( f(x) = \sqrt{x} - \ln(x) \)
   a) The graph of \( f \) has an asymptote. What is it? (1)
   b) Construct a sign chart for \( f'(x) \) and then give the intervals where \( f \) is increasing and the intervals where \( f \) is decreasing. Give both coordinates of any relative extrema. (3)
   c) Construct a sign chart for \( f''(x) \) and then give the intervals where \( f \) is concave up and the intervals where \( f \) is concave down. Give both coordinates of any inflection points. (3)
   d) Accurately draw the graph of \( f \) that shows all features found in parts a), b) and c). (3)

17. In the diagram above, a right triangle with sides \( x \) and \( y \) has squares adjoined to its three sides. If \( x + y = 10 \), what is the minimum possible sum of the areas of the three squares and the triangle? (4)

18. Verify that the conditions of the Mean Value Theorem are satisfied for the function \( f(x) = \frac{x^2 - 2}{x - 1} \) on the interval \([2, 4]\) and then find all the values of \( c \) that satisfy the conclusion. (4)

**Total = 121**