COLUMBIA COLLEGE
Mathematics 113
Final Examination, December 12, 2005

Please do not have any of the following in your possession during the examination: notes, books, pencil cases, graphics calculators, electronic dictionaries or cell phones.

1. a) State the informal definition of \( \lim_{x \to a} f(x) = L \) \( \hfill \) (1)
b) State the formal ("\( \epsilon, \delta \)") definition of \( \lim_{x \to a} f(x) = L \) \( \hfill \) (1)
c) Give an "\( \epsilon, \delta \)" proof for \( \lim_{x \to 1} 3x + 1 = 4 \). \( \hfill \) (3)
d) Give an "\( \epsilon, \delta \)" proof for \( \lim_{x \to 1} \frac{1}{x} = 2 \). \( \hfill \) (3)

2. Suppose that \( f(x) = 2x + 1 \) if \( x < 1 \) and that \( f(x) = x^2 + 1 \) if \( x \geq 1 \).
a) Prove that \( f \) is continuous at \( x = 1 \) by verifying that the three conditions of continuity at a point are satisfied. \( \hfill \) (3)
b) Using two theorems, prove that \( f \) is continuous at all other values of \( x \). \( \hfill \) (2)

3. Without using L’Hôpital’s rule, find the following limits. (3 marks each)
   a) \( \lim_{x \to 1} \frac{x^3 + x - 2}{x^2 - 1} \)
   b) \( \lim_{x \to 1} \frac{3 - \sqrt{x + 8}}{2x - 2} \)
   c) \( \lim_{x \to \infty} \frac{\sqrt[3]{8x^6 + 4x^4}}{x^2 + 3x + 1} \)
   d) \( \lim_{x \to 0} \frac{\sin x}{x} \)

4. A particle moves along a line so that at time \( t \) seconds its position is \( x = 2t^2 + t \) centimeters from the start (\( t = 0 \))
a) What is the average velocity of the particle in the interval \( 2 \leq t \leq 4 \)? \( \hfill \) (1)
b) What is the instantaneous velocity of the particle at \( t = 3 \)? \( \hfill \) (2)

5. Using only the definition of the derivative function, \( f'(x) \), find \( f'(x) \) for the function \( f(x) = -3x^2 + 2x - 1 \). \( \hfill \) (3)

6. Suppose that \( f(x) = -x \) if \( x < 1 \) and that \( f(x) = -x^2 + 4x - 4 \) if \( x \geq 1 \).
a) Prove that \( f \) is not differentiable at \( x = 1 \) by proving that the two-sided limit \( f'(1) \) does not exist. \( \hfill \) (3)
b) Accurately draw the graph of \( f \) on the interval \([-1, 1]\) and then explain how this graph shows that \( f \) is not differentiable at \( x = 1 \). \( \hfill \) (2)
7. Find the slope-intercept equation of the tangent line to the curve 
\[ y = \frac{\sqrt{x + 3}}{x^2} \] at the point (1,2).  

8. If \( f(x) = x^2 \sin 2x \), find the exact value of \( f''(\pi/6) \).

9. A point \( P \) is moving counterclockwise around the circle \( x^2 + y^2 = 4 \) so that the angle \( \theta \) shown in the diagram above is changing at a rate of 1 radian per second. At what rate is the y-coordinate of \( P \) changing when \( \theta = \pi/4 \)?

10. Find all the points on the curve \( y = \cos 2x - 4 \sin x \) where the tangent line is horizontal.

11. Given the relation \( e^x + y + x^2 y + x = 1 \), express \( \frac{dy}{dx} \) in terms of \( x \) and \( y \) and find the value of \( \frac{dy}{dx} \) at the point (1,-1).

12. Using logarithmic differentiation, find \( y' \) for \( y = \frac{(x^2 + 2)^3 \sqrt{x^2 + 3}}{\sin(x^4)} \)

13. Let \( f(x) = \frac{e^x + 1}{e^x + 2} \)
   a) Prove that \( f \) is a one-to-one function.
   b) Using the theorem about the derivative of an inverse function, find the value of the derivative of \( f^{-1}(x) \) when \( x = \frac{1}{2} \).
   c) Find a formula for \( f^{-1}(x) \) and then find the answer to b) directly.
14. Use L’Hôpital’s rule to find:

d) \( \lim_{x \to 0} \frac{\ln(\cos x)}{1 - \cos x} \) \hspace{2cm} (3)

d) \( \lim_{x \to 0} (1 - 2x)^{4/x} \) \hspace{2cm} (4)

15. Let \( \sqrt{x} - \frac{1}{2} \ln x \)

a) Give the domain of \( f \) and all the intercepts and asymptotes of the graph. \( (2) \)

b) Find the intervals where \( f \) is increasing, the intervals where \( f \) is decreasing and both coordinates of any relative extrema. \( (2) \)

c) Find the intervals where \( f \) is concave up, the intervals where \( f \) is concave down and both coordinates of any inflection points. \( (2) \)

d) Accurately draw the graph of \( f \) showing all these features. \( (2) \)

16. Let \( f(x) = \frac{1}{3} x + x^{2/3} \)

a) Explain why \( f \) must have both an absolute maximum and an absolute minimum on any given closed interval \([a,b]\) and explain how these values may be found. \( (2) \)

b) Find the absolute extrema of \( f \) on the interval \([-10, \frac{1}{2}]\). \( (3) \)

17. Use Newton’s method to calculate the zero of \( f(x) = x^3 + x - 80 \) rounded to two decimal places. You must justify your choice for the first approximation, \( x_0 \). \( (3) \)

18. Let \( f(x) = \sin^{-1}(2x) \). Verify that the hypotheses of the Mean Value Theorem are satisfied by \( f \) on the interval \([0, \frac{1}{2}]\) and find all the values of \( c \) that satisfy the conclusion of the theorem. \( (4) \)
19. Let \((x,0)\) be the lower left corner of the rectangle inscribed in triangle ABC shown above. Find the value of \(x\) that maximizes the area of the rectangle.

20. a) Theorem: If \(F(x)\) and \(G(x)\) are both antiderivatives of \(f(x)\) on \((a,b)\) then _______.

b) Find \(\int \left(5x^2 + \frac{4}{x} + \sin(2x) + e^{3x}\right) \, dx\)

c) Solve the initial value problem: \(\frac{dy}{dx} = \sqrt{2x+1}\), \(y(4) = 0\).

Total = 100