COLUMBIA COLLEGE
MATHEMATICS 113
Final Examination (3 hours)
August 8, 2005

1. a) (Informal) Definition: \( \lim_{x \to a} f(x) = L \) means that as \( x \) approaches \( a \) but the values of \( f(x) \) approach \( L \). (3)
b) (Informal) Definition: \( \lim_{x \to \infty} f(x) = L \) means \( \) (2)
c) Definition: \( f(x) \) is continuous at \( x = a \) if: 1) \( \) (3)
and 3) \( \)
d) (Formal - “\( \varepsilon, \delta \)”) Definition: \( \lim_{x \to a} f(x) = L \) if \( \) (2)

2. Find the following limits (possible answers: a real number, \( \infty, -\infty \) or “does not exist”) Show work and give explanations. Do not use L'Hopital's rule. (3 marks each)
   a) \( \lim_{x \to 5} \frac{x^3 + x - 130}{x^2 - 25} \)
   b) \( \lim_{x \to 4} \frac{4 - \sqrt{x} + 12}{x - 4} \)
   c) \( \lim_{x \to 4^-} \frac{x - 3}{x^2 + x - 20} \)
   d) \( \lim_{x \to \infty} x - \sqrt{x^2 + 8x} \)
   e) \( \lim_{x \to 0} \frac{\sin 2x}{1 - e^x} \)

3. Suppose that the function \( f \) is defined by two formulas: if \( x \leq 2 \) then \( f(x) = 4x - 1 \) and if \( x > 2 \) then \( f(x) = x^2 + 3 \). Prove that \( f \) is continuous at \( x = 2 \) (Verify that three conditions have been satisfied. Also explain why \( f \) is continuous at all other values of \( x \). (5)

5. Give an “\( \varepsilon, \delta \)” proof that \( \lim_{x \to 4} x^2 + 2x = 24 \) (5)

6. A particle moves in the positive direction along a line so that after \( t \) minutes its distance from the origin is \( s(t) = 4t^2 + 3t + 2 \) feet.
   a) Find the average velocity of the particle over the time interval \([1,3]\). (2)
   b) By calculating a limit, find the (instantaneous) velocity at \( t = 2 \). (2)

7. Using only the definition of the derivative function, find the derivative of \( f(x) = x^2 + \frac{2}{x} \) (5)
8. Find the slope-intercept equation of the line that is tangent to the curve 
\[ y = \frac{x^2 + \sqrt{x}}{x + \sqrt{x}} \] at the point (1,1). (5)

9. Use the local linear approximation of \( f(x) = \log_2(x) \) at \( x_0 = 2 \) to approximate \( f(2.01) \). Express your answer rounded to six decimal places. (5)

10. Use logarithmic differentiation to find the derivative of 
\[ y = \frac{(x^2 + 1)^5(x^3 + x)^2}{\sqrt{x^2 + 3x}} \] (5)

11. Use L'Hôpital's rule to find: 
   a) \( \lim_{x \to 0} \frac{2x - \sin 2x}{x^3} \) 
   b) \( \lim_{x \to \infty} \frac{\ln x}{\csc x} \) (5)

12. a) State the Mean Value Theorem (MVT) (1)
b) Verify that the conditions of the MVT are satisfied by the function 
\( f(x) = x^3 + 2x^2 \) on the interval [2,6]. (2)
c) Find the number \( c \) that satisfies the conclusion of the MVT for \( f(x) = x^3 + 2x^2 \) on the interval [2,6]. (3)

13. Accurately draw a function \( f(x) \) that has all the following properties:
   - \( f \) is continuous on \([0,4)\) and \((4,\infty)\)
   - \( f(0) = f(3) = 1 \)
   - On \((0,1)\): \( f'(x) < 0, f''(x) < 0 \)
   - On \((1,2)\): \( f'(x) > 0, f''(x) < 0 \)
   - \( f''(2) = 0 \) and \((2,0)\) is an inflection point
   - \( \lim_{x \to 4^-} f(x) = -\infty, \lim_{x \to 4^+} f(x) = -\infty, \lim_{x \to \infty} = 1 \) (5)

14. A particle is moving along the curve \( y = \frac{x}{x^2 + 1} \). Find all values of \( x \) such that the rate that \( x \) is changing with respect to \( t \) (time) is three times the rate that \( y \) is changing with respect to \( t \). (5)

15. For the relation \( y^3 + 2xy^2 - 3x^2y + x^3 = 1 \), use implicit differentiation to find \( \frac{dy}{dx} \) as a function of \( x \) and \( y \) and then find the slope-intercept equation of the line that is tangent to the graph of the relation at the point \((1,1)\). (5)
16. Accurately draw the graph of \( f(x) = \frac{1}{x} - \frac{1}{x^2} \) showing all features of interest and the analyses using the first and second derivatives. (8)

17. Accurately draw the graph of \( \sqrt[3]{x^2 - 4} \) showing all features of interest and the analyses using the first and second derivatives. (Be careful at the critical points) (8)

18. In the diagram above AB = CD. Find the area of the largest such L-shape. (8)

19. a) Accurately draw the graphs of \( y = e^x \) and \( y = 2 - x \) in the same coordinate grid over the interval [-1,2] to determine a good estimate, \( x_0 \) for the zero of the function

\[
f(x) = e^x + x - 2.
\]

b) Use Newton’s method to calculate this zero rounded to four decimal places. (4)

20. Evaluate the following indefinite integrals: (3 marks each)

\[
a) \int \frac{1}{2\sqrt{1 - x^2}} + \frac{2}{3x} \, dx \quad b) \int \csc(x)(\sin x + \cot x) \, dx
\]

END