COLUMBIA COLLEGE
Mathematics 111
Final Examination, December 10, 2005

Please do not have any of the following in your possession during the examination: notes, books, pencil cases, graphics calculators, electronic dictionaries or cell phones.

1. Let \( f(x) \) be defined using two formulas:

If \( x \leq 3 \) then \( f(x) = x^2 - 7 \) and if \( x > 3 \) then \( f(x) = \frac{x^2 + 1}{x + 2} \)

a) Prove that \( f \) is continuous at \( x = 3 \) by showing that all three conditions of the definition of continuity at a point are satisfied. (3)
b) Prove, using theorems, that \( f \) is continuous at all other values of \( x \). (2)

2. Given:

Limit Theorems:
Suppose that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist and that \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \) then:

1. \( \lim_{x \to a} (f(x) \pm g(x)) = F \pm G \)
2. \( \lim_{x \to a} f(x)g(x) = FG \)
3. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G} \) provided that \( G \neq 0 \)
4. \( \lim_{x \to a} cf(x) = cF \) for all constants \( c \)
5. \( \lim_{x \to a} c = c \)
6. \( \lim_{x \to a} x^n = a^n \) for any real exponent \( n \)

Given that \( \lim_{x \to 3} f(x) = 18 \), use these limit theorems to prove that

\[
\lim_{x \to 3} \frac{(3x + 1)f(x)}{2x^2} = 10.
\]

The theorems must be used in a logical order and may be referred to by number. (5)
3. Compute the following limits. Show all work. (3 marks each)
   
   a) \( \lim_{x \to 3} \frac{3-x}{3-\sqrt{x}+6} \)  
   b) \( \lim_{x \to 3} \frac{x^4 - 81}{x^2 - x - 6} \)  
   c) \( \lim_{x \to \infty} \frac{100 + \sqrt{25x^8 + 7}}{10 + x^4} \)  
   d) \( \lim_{n \to \infty} \left( 1 + \frac{4}{5n} \right)^{-2n} \)

4. Let \( f(x) = \ln x + e^x \)
   
   a) State the Intermediate Value Theorem and then use this theorem to prove that there exists a value of \( x \) in \([1,2]\) such that \( f(x) = 4 \). (2)

   b) Calculate this value of \( x \) rounded to one decimal place. You must explain the method you used to get your answer. (2)

5. Using the definition of the slope of the tangent as a limit, compute the slope of the tangent line to the curve \( y = 2x^2 + 3x \) at the point \((2,14)\) and then find the slope-intercept equation of this tangent line. (5)

6. Use the Central Difference Formula with \( \Delta x = 0.1 \) to approximate \( f'(3) \) for the function \( f(x) = \ln x \) and then compare with actual value of \( f'(3) \) rounded to six decimal places. (3)

7. A small company that produces and sells dog-houses has fixed costs of \$1000 per week. In addition, each bird-house costs \$9.50 to make. The weekly revenue is given by the formula \( R(x) = 12x - 0.01x^2 \) for \( 0 \leq x \leq 100 \) where \( x \) is the number of dog-houses that are produced and sold each week. Find the marginal cost, the marginal revenue and the marginal profit. (You must use the appropriate units in your answer.) (3)

8. Use linear approximation with \( f(x) = x^{-1/4} \) and an appropriate value of \( a \) to approximate \( \frac{1}{\sqrt{79}} \) and compare with the actual value rounded to six decimal places. (5)
9. Let \( f(x) = x^2 + 2x \) if \( x \leq 1 \) and \( f(x) = 4x - 1 \) if \( x > 1 \). Use the definition of the derivative to show that \( f \) is not differentiable at \( x = 1 \). (5)

10. If the inflation rate is 8% per year over a period of years, an object costing $100 today will cost \( 100(1.08^t) \) dollars in \( t \) years. How much will the object cost and how fast will its price be rising 8 years from today? (Use appropriate units in your answer.) (3)

11. Find the points on the graph of \( f(x) = (2 + \frac{1}{x})e^x \) where the tangent line is horizontal. Give both coordinates of these points. (5)

12. Find the slope-intercept equation of the line that is tangent to the curve \( y = \frac{x \ln x}{(1 + \ln x)^2} \) at the point \((e, e/4)\). (5)

13. Given the relation \( 2x^3 + 4xy^2 + y^3 = 5 \), find \( \frac{dy}{dx} \) as a function of \( x \) and \( y \) and then find the value of \( \frac{dy}{dx} \) at the point \((1, -1)\). (5)

14. In the diagram above the height of a falling ball at time \( t \) seconds is \( h = 64 - 16t^2 \) feet. At what rate is the distance from point A to the ball changing when \( h = 48 \) feet? (5)
15. For \( f(x) = x^4 - 5x^2 + 4 \)
   a) Find the intercepts, symmetry and asymptotes \( \quad (2) \)
   b) Find the intervals where \( f \) is increasing and where \( f \) is decreasing and give both coordinates of any local extrema. \( \quad (2) \)
   c) Find the intervals where \( f \) is concave up and where \( f \) is concave down and give both coordinates of any inflection points. \( \quad (2) \)
   d) Accurately draw the graph of \( f(x) \) showing all these features. \( \quad (4) \)

16. Use Newton's method to find the root of \( e^x + x = 50 \) rounded to two decimal places. Show all work. \( \quad (5) \)

17. Let \( f(x) = (x^2 + x)^{2/3} \)
   a) Explain why \( f \) must have both a global maximum and a global minimum on any given closed interval \([a,b]\) and explain how to find these values. \( \quad (2) \)
   b) Find the global extrema of \( f \) on \([-2,2]\). \( \quad (3) \)

18. A rectangle is inscribed in the top half of the ellipse \( \frac{x^2}{4} + y^2 = 1 \) as shown in the diagram above. \((x,0)\) is the lower right vertex of the rectangle. Find the maximum area of such a rectangle. \( \quad (5) \)

19. a) State the definition of the indefinite integral \( \int f(x) \, dx \). \( \quad (1) \)
   b) Find \( \int 2x^2 + \frac{1}{3x} + e^{2x} \, dx \) \( \quad (2) \)
   c) Solve the initial value problem: \( \frac{dy}{dx} = x^3 + x^2 + 1 \), \( y(1) = 2 \). \( \quad (2) \)

Total = 100